Chapter 1

Discrete Models

1.1 Discrete Models

Suppose[1] that a time series of $q+1$ data points

$$y_0, y_1, y_2, \ldots, y_q$$

is given. A *likelihood function* L gives the probability that the observed data would result from the proposed stochastic mechanism relative to all other possible outcomes [132]. The data y_t is a realization of the random variable $x(t)$. On the log scale, $w_t = \ln y_t$ is a realization of the random variable $\ln x(t)$. The likelihood function L is

$$L(\theta_1, \ldots, \theta_p, v) = \prod_{t=1}^{q} p(w_t|w_{t-1}),$$

where $p(w_t|w_{t-1})$ is the joint probability distribution function (pdf) that w_t occurs given that w_{t-1} occurs. This is a normal pdf with mean $\ln f(y_{t-1}, \theta_1, \ldots, \theta_p)$ and variance v. Thus,

$$p(w_t|w_{t-1}) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{2v}\left(w_t - \ln f(y_{t-1}, \theta_1, \ldots, \theta_p)\right)^2\right)$$

and

$$L(\theta_1, \ldots, \theta_p, v) = \prod_{t=1}^{q} \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{2v}\left(w_t - \ln f(y_{t-1}, \theta_1, \ldots, \theta_p)\right)^2\right).$$

The *maximum likelihood parameter estimates* are those values of the parameters $\theta_1, \ldots, \theta_p, v$ that maximize $L(\theta_1, \ldots, \theta_p, v)$, or equivalently that maximize

[1] Sample text from *An Introduction to Structured Population Dynamics* by J. M. Cushing, CBMS-NSFT Regional Conference Series in Applied Mathematics.

$l(\theta_1,\ldots,\theta_p,v) \doteq \ln(L(\theta_1,\ldots,\theta_p,v))$. A calculation shows

$$l(\theta_1,\ldots,\theta_p,v) = -\frac{q}{2}\ln(2\pi) - \frac{q}{2}\ln v - \frac{1}{2v}\sum_{t=1}^{q} r_t^2(\theta_1,\ldots,\theta_p),$$

where

$$r_t(\theta_1,\ldots,\theta_p) \doteq \ln y_t - \ln f(y_{t-1},\theta_1,\ldots,\theta_p) = \ln\left(\frac{y_t}{f(y_{t-1},\theta_1,\ldots,\theta_p)}\right)$$

are the log-residuals. The critical points $(\theta_1,\ldots,\theta_p,v)$ of l are zeroes of the derivatives

$$\partial_{\theta_i} l = -\frac{1}{v}\sum_{t=1}^{q} r_t(\theta_1,\ldots,\theta_p)\,\partial_{\theta_i} r_t(\theta_1,\ldots,\theta_p),$$

$$\partial_v l = -\frac{q}{2}\frac{1}{v} + \frac{1}{2v^2}\sum_{t=1}^{q} r_t^2(\theta_1,\ldots,\theta_p),$$

i.e., are roots of the uncoupled equations

$$\sum_{t=1}^{q} r_t(\theta_1,\ldots,\theta_p)\,\frac{\partial_{\theta_i} f(y_{t-1},\theta_1,\ldots,\theta_p)}{f(y_{t-1},\theta_1,\ldots,\theta_p)} = 0,$$